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The heated laminar vertical jet

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The boundary-layer equations for the steady laminar flow of a vertical jet, including a buoyancy term caused by temperature differences, are solved by similarity methods. Two-dimensional and axisymmetric jets are treated. Exact solutions in closed form are found for certain values of the Prandtl number, and the velocity and temperature distribution for other Prandtl numbers are found by numerical integration.

1. Introduction

The problem investigated here is that of a heated fluid jet flowing vertically upward from an orifice into a region of the same fluid, which, apart from the influence of the jet, is at rest and at a uniform temperature. The flow through the orifice is produced by a pressure difference, and immediately upon emergence the fluid in the jet is subjected to a buoyant force which arises as a result of the difference in temperature between the jet and the surrounding fluid. Both the two-dimensional case, representing a jet emerging from a long narrow slit, and the axisymmetric case, representing flow from a circular opening are considered.

The flow is assumed to be steady and laminar, with the variation of density small enough to permit the use of the incompressible equations of motion modified only by the inclusion of a buoyancy term. It is also assumed that the rates of change of velocity components and temperature in the streamwise direction are much less than in the transverse directions. Thus the boundary-layer approximations to the momentum and energy equations apply. Frictional heating of the fluid is neglected.

These assumptions require that the Reynolds and Grashof numbers, based on a transverse dimension of the jet, be of the same order of magnitude and much larger than unity. The product of the volume coefficient of expansion and the mean temperature difference must be small. The assumptions and the consequent requirements on the physical characteristics of the jet are discussed in more detail in §4.

The free jet problem, without the effect of buoyancy, was solved by Schlichting (1933) and by Bickley (1937) by methods very similar to that employed here, and experimental confirmation was obtained by Andrade (1939). The differential equations for the present problem are the same as those for natural convection along a vertical hot plate, a problem studied in considerable detail by Sparrow & Gregg (1958).

2. The two-dimensional jet

A system of rectangular co-ordinates (x, y) is chosen such that the x-axis coincides with the symmetry axis of the jet. The boundary-layer equations, expressing the conservation laws of mass, momentum, and energy are:

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta T_\infty \theta, \tag{2.2}$$

$$\sigma\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\right) = \nu\frac{\partial^2 \theta}{\partial y^2}. \tag{2.3}$$

In these equations, u and v are velocity components, and θ is a dimensionless temperature difference related to the local temperature, $T(x, y)$, and to the temperature far from the jet, T_∞, by

$$\theta = (T - T_\infty)/T_\infty. \tag{2.4}$$

Other constants appearing in the equations are: ν, the kinematic viscosity; β, the coefficient of volume expansion; g, the acceleration of gravity; and σ, the Prandtl number.

Boundary conditions to be satisfied are:

$$\text{at} \quad y = 0, \quad v = \partial u/\partial y = \partial \theta/\partial y = 0; \tag{2.5}$$

$$\text{at} \quad y = \infty, \quad u = \theta = 0. \tag{2.6}$$

The continuity equation (2.1) implies the existence of a stream function, $\psi(x, y)$, such that

$$u = \partial \psi/\partial y, \quad v = -\partial \psi/\partial x. \tag{2.7}$$

The partial differential equations are reduced to ordinary equations by means of the following transformation:

$$\eta = a y x^{\alpha - 1}, \tag{2.8}$$

$$\psi = a \nu x^{\alpha} f(\eta), \tag{2.9}$$

$$\theta = a^4 \nu^2 (g\beta T_\infty)^{-1} x^{4\alpha - 3} p(\eta). \tag{2.10}$$

The arbitrary constant, a, is included so as to make $\eta, f(\eta)$ and $p(\eta)$ dimensionless. It can be chosen so as to match the mathematical solution to a particular physical case. The exponent, α, is, in a sense, an eigenvalue, in that non-trivial solutions of the equations which satisfy the boundary conditions exist only for a specific value of the exponent.

When the momentum and energy equations are transformed according to (2.7), (2.8), (2.9) and (2.10), there results:

$$f''' + \alpha f f'' - (2\alpha - 1)(f')^2 + p = 0, \tag{2.11}$$

$$p'' + \alpha \sigma f p' + (3 - 4\alpha) \sigma f' p = 0. \tag{2.12}$$

Boundary conditions in terms of f and p are

$$f(0) = f''(0) = p'(0) = 0, \tag{2.13}$$

$$f'(\infty) = p(\infty) = 0. \tag{2.14}$$

The constant α is determined by an integration of the energy equation (2.12), which may be written

$$\frac{1}{\sigma}\frac{dp'}{d\eta}+\alpha\frac{d}{d\eta}(fp)+(3-5\alpha)f'p = 0. \tag{2.15}$$

Because of the boundary conditions, (2.13) and (2.14), integration of (2.15) over the interval $(0, \infty)$ gives

$$(3-5\alpha)\int_0^\infty f'p\,d\eta = 0. \tag{2.16}$$

Since p and f' are never negative, the only way in which (2.16) can be satisfied is

$$\alpha = \tfrac{3}{5}. \tag{2.17}$$

With this value of α, equation (2.12) can be integrated once, and the governing equations for the problem become

$$f'''+\tfrac{3}{5}ff''-\tfrac{1}{5}(f')^2+p = 0, \tag{2.18}$$

$$p'+\tfrac{3}{5}\sigma fp = 0. \tag{2.19}$$

The velocity components and the temperature are related to f and p by

$$u = a^2\nu x^{\frac{1}{5}}f', \tag{2.20}$$

$$v = \tfrac{1}{5}a\nu x^{-\frac{2}{5}}(2\eta f'-3f), \tag{2.21}$$

$$\theta = a^4\nu^2(g\beta T_\infty)^{-1}x^{-\frac{3}{5}}p. \tag{2.22}$$

3. Exact and numerical solutions

Exact solutions for the system (2.18), (2.19), which satisfy the boundary conditions (2.13) and (2.14), have been found for $\sigma = 2$ and $\sigma = \frac{5}{9}$. If p is assumed related to f by

$$p = \tfrac{4}{5}(f')^2+b[(f')^2+ff''], \tag{3.1}$$

equation (2.18) becomes

$$f'''+(\tfrac{3}{5}+b)ff''+(\tfrac{3}{5}+b)(f')^2 = 0, \tag{3.2}$$

which has the integral

$$f''+(\tfrac{3}{5}+b)ff' = 0. \tag{3.3}$$

The same assumption for p, equation (3.1), is now substituted in the energy equation (2.19), with the result,

$$bf[f'''+\tfrac{3}{5}\sigma ff''+\tfrac{3}{5}\sigma(f')^2]+(\tfrac{8}{5}+3b)f'\left[f''+\frac{12\sigma}{40+75b}ff'\right] = 0. \tag{3.4}$$

Inspection of this equation shows that for certain values of b and σ any function which satisfies (3.3) also satisfies (3.4). The appropriate values of b and σ are:

case (i) $b = 0, \quad \sigma = 2;$

case (ii) $b = -\frac{4}{15}, \quad \sigma = \frac{5}{9}.$

In case (i), the coefficient of the first bracketed expression in (3.4) vanishes, and the second bracketed expression becomes identical with (3.3), which is, in this case,

$$f''+\tfrac{3}{5}ff' = 0. \tag{3.5}$$

The solution of (3.5) and the corresponding expression for p from (3.1) are

$$f = \tanh\left(\tfrac{3}{10}\eta\right), \tag{3.6}$$

$$p = \tfrac{9}{125}\operatorname{sech}^4\left(\tfrac{3}{10}\eta\right). \tag{3.7}$$

In case (ii), the bracketed expressions in (3.4) become identical to the left sides of (3.2) and (3.3). The solution for this case is

$$f = \tanh\left(\tfrac{1}{6}\eta\right), \tag{3.8}$$

$$p = \tfrac{2}{135}\operatorname{sech}^2\left(\tfrac{1}{6}\eta\right). \tag{3.9}$$

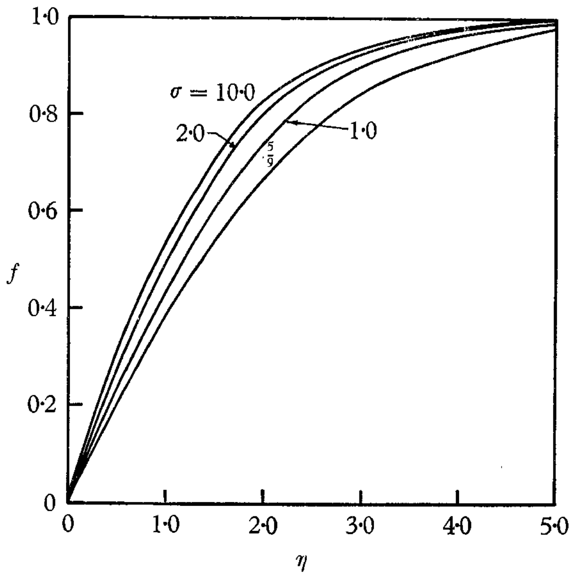

FIGURE 1. Two-dimensional jet. f *vs.* η for several values of σ.

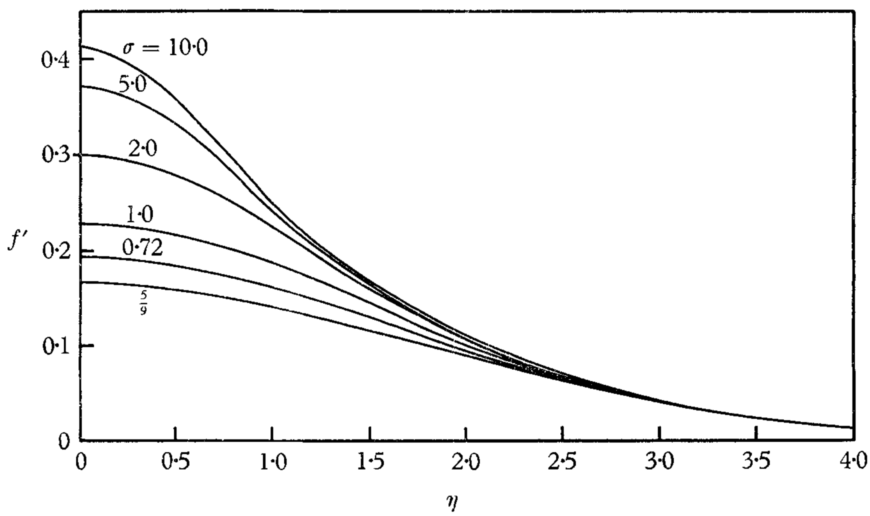

FIGURE 2. Two-dimensional jet. f' *vs.* η for several values of σ.

Numerical integrations of equations (2.18) and (2.19), subject to boundary conditions (2.13) and (2.14), were carried out for several values of the Prandtl number. A forward integration scheme employing Runge–Kutta fourth-order formulas was used. The solutions are shown in figures 1–4.

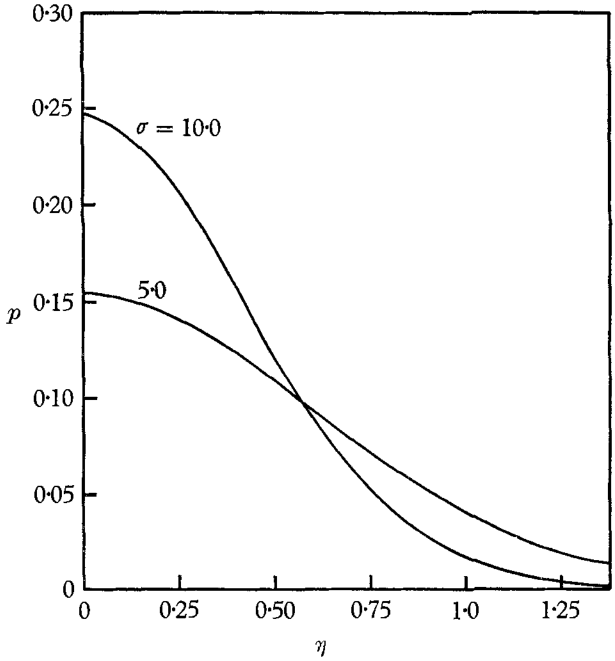

FIGURE 3. Two-dimensional jet. p *vs.* η for two values of σ.

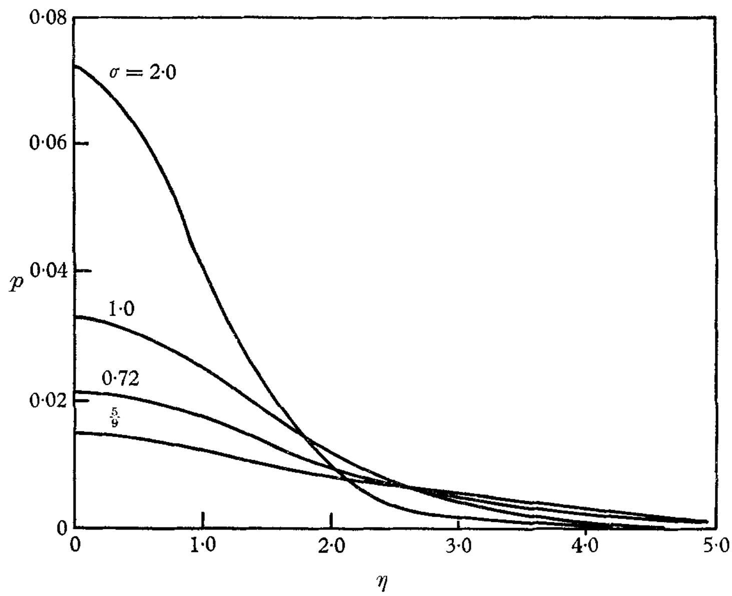

FIGURE 4. Two-dimensional jet. p *vs.* η for several values of σ.

The value of $f'(0)$ used at the outset of the numerical integration is arbitrary, and corresponding to each choice of $f'(0)$ there is a resulting value of $f(\infty)$. That solution was chosen which resulted in $f(\infty) = 1$.

4. Interpretation of the solution

From the solutions for f, f' and p, one may compute the velocity components and the temperature from equations (2.20), (2.21) and (2.22). However, the constant, a, and the location of the origin of x have not as yet been specified. Both of these quantities are related to the initial conditions of the jet, and can be chosen so as to match the mathematical solution to a particular physical experiment.

Suppose that at some station, the velocity and the temperature profiles are measured. Then at some x, say $x = x_0$, the mathematical solution must match the measured profiles, at least in an average sense. For instance, if W is the measured volume flow rate, and E is the integral of the measured temperature profile (and

σ	$\int_0^\infty p\,d\eta$
$\frac{5}{9}$	0·0367
0·72	0·0452
1·0	0·0575
2·0	0·0888
5·0	0·1124
10·0	0·1335

TABLE 1

is thus a measure of the thermal energy in the jet), one can select a and x_0 so as to make the corresponding integrals of the mathematical solution equal to the measured quantities,

$$2\int_0^\infty u(x_0, y)\,dy = W, \tag{4.1}$$

$$2\int_0^\infty [T(x_0, y) - T_\infty]\,dy = E. \tag{4.2}$$

When equations (2.20) and (2.22) are substituted in (4.1) and (4.2), the results may be solved for a and x_0:

$$2ax_0^{\frac{3}{5}} = W/\nu, \tag{4.3}$$

$$2a = \left[W(4g\beta E)^3\nu^{-7}\left(\int_0^\infty p\,d\eta\right)^{-3}\right]^{\frac{1}{10}}, \tag{4.4}$$

$$2x_0 = \left[W^3\left(\int_0^\infty p\,d\eta\right)(\nu g\beta E)^{-1}\right]^{\frac{1}{2}}. \tag{4.5}$$

The value of x_0 locates the origin of the co-ordinate system, and the mathematical solutions may be assumed to represent the actual jet flow accurately for $x \geqslant x_0$.

Values of the integral in equations (4.4) and (4.5) are presented in table 1 for several Prandtl numbers.

It should be noted that the centre-line velocity increases with x,

$$u(x,0) \sim x^{\frac{1}{5}}, \tag{4.6}$$

as contrasted with the isothermal jet, in which

$$u(x,0) \sim x^{-\frac{1}{3}}. \tag{4.7}$$

This implies that the buoyant force more than balances the viscous drag and produces an upward acceleration of the fluid. However, if the amount of heating is small, E is small, and x_0 is large. Since the solution is valid only for $x \geqslant x_0$, the smaller the amount of heating the smaller the acceleration.

The validity of the assumptions may be checked by comparing the orders of magnitude of various terms. In order to test the boundary-layer assumption, one compares $\partial u/\partial x$ and $\partial u/\partial y$ and finds

$$\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y} = \frac{2\nu}{5W}\left(\frac{x}{x_0}\right)^{-\frac{3}{5}}\left(\frac{f'}{f''}-2\eta\right). \tag{4.8}$$

If the width of the jet at $x = x_0$ is $2h$, and the mean velocity at this station is U_0, then

$$W = 2U_0 h, \tag{4.9}$$

and

$$W/2\nu = U_0 h/\nu = R. \tag{4.10}$$

Hence, if $R \gg 1$, $\partial u/\partial x \ll \partial u/\partial y$, except at the centre line where f'' vanishes.

One also would like assurance that the similarity solution is not too different from the measured jet profile at $x = x_0$. For this to be true the value of η at $y = h$, $x = x_0$, must be such that the fluid velocity as given by the similarity solution at this point is much less than at the centre line. From figures 2 and 4 it is seen that, at least for the larger Prandtl numbers, $f' \ll f'(0)$ and $p \ll p(0)$ when $\eta \geqslant 4$. Let us take, then

$$\eta(x_0, h) \geqslant 4. \tag{4.11}$$

This leads to

$$(G/R)^{\frac{1}{2}} \geqslant \left[\int 8p\,d\eta\right]^{\frac{1}{2}}, \tag{4.12}$$

where G is the Grashof number,

$$G = g\beta\Delta T_0 h^3 \nu^{-2}, \tag{4.13}$$

and ΔT_0 is the mean temperature difference at the location $x = x_0$. The result (4.12) indicates that the Grashof and Reynolds numbers must be comparable.

Justification of the neglect of compressibility is based on an examination of the continuity equation, which for a compressible fluid may be written

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{1}{\rho}\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right) = 0. \tag{4.14}$$

A comparison of the neglected terms with those retained leads to the conclusion that for (2.1) to be a valid approximation to (4.14), it is necessary for

$$\beta\Delta T_0 \ll 1. \tag{4.15}$$

The requirements that $\beta\Delta T_0$ be small and G be large are not contradictory, as may be seen by a numerical example. If $\beta\Delta T_0$ is of order 10^{-2} and G is of order 10^2, the jet width, h, must be a few tenths of an inch, if the fluid is air at room temperature.

5. The axisymmetric jet

Analysis of the axisymmetric heated jet is completely analogous to the two-dimensional problem. The x-axis is now the jet axis, and y is the radial co-ordinate. With the other quantities defined as before, the boundary-layer equations are:

$$\frac{\partial(uy)}{\partial x}+\frac{\partial(vy)}{\partial y} = 0, \tag{5.1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = \nu\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right)+g\beta T_\infty\theta, \tag{5.2}$$

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y} = \frac{\nu}{\sigma}\frac{1}{y}\frac{\partial}{\partial y}\left(y\frac{\partial\theta}{\partial y}\right). \tag{5.3}$$

Boundary conditions to be satisfied are:

$$\text{at}\quad y=0;\quad v=\frac{\partial u}{\partial y}=\frac{\partial\theta}{\partial y}=0;\quad u \text{ is finite;}$$

$$\text{at}\quad y=\infty;\quad u=\theta=0.$$

The continuity equation is again integrated by means of a stream function,

$$u=\frac{1}{y}\frac{\partial\psi}{\partial y},\quad v=-\frac{1}{y}\frac{\partial\psi}{\partial x}, \tag{5.4}$$

and introduction of a similarity transformation,

$$\eta = ayx^\alpha, \tag{5.5}$$

$$\psi = \nu x f(\eta), \tag{5.6}$$

$$\theta = \frac{a^4\nu^2}{g\beta T_\infty}x^{1+4\alpha}p(\eta), \tag{5.7}$$

leads to the ordinary differential equations

$$d/d\eta(f''-f'/\eta)+ff''/\eta-ff'/\eta^2-(1+2\alpha)(f')^2/\eta+\eta p = 0, \tag{5.8}$$

$$1/\sigma(p''+p'/\eta)+fp'/\eta-(1+4\alpha)f'p/\eta = 0. \tag{5.9}$$

Boundary conditions on f and p are:

$$f(0)=f'(0)=p'(0)=0, \tag{5.10}$$

$$p(0) \text{ is finite}, \tag{5.11}$$

$$p(\infty) = \lim_{\eta\to\infty}[f'(\eta)/\eta] = 0. \tag{5.12}$$

The similarity exponent, α, is determined by integrating the energy equation (5.9), which may be written

$$d/d\eta(\eta p'/\sigma+fp)-(2+4\alpha)f'p = 0. \tag{5.13}$$

Integration from zero to infinity and application of the boundary conditions give

$$(2+4\alpha)\int_0^\infty f'p\,d\eta = 0,$$

which can be satisfied only by taking

$$\alpha = -\tfrac{1}{2}. \tag{5.14}$$

Equation (5.13) now has a first integral, and the governing equations become

$$d/d\eta(f''-f'/\eta)+ff''/\eta-ff'/\eta^2+\eta p = 0, \tag{5.15}$$

$$\eta p'+\sigma f p = 0. \tag{5.16}$$

The velocity components in terms of f are

$$u = a^2\nu f'/\eta, \tag{5.17}$$

$$v = a\nu x^{-\frac{1}{2}}[\tfrac{1}{2}f'+(f/\eta)]. \tag{5.18}$$

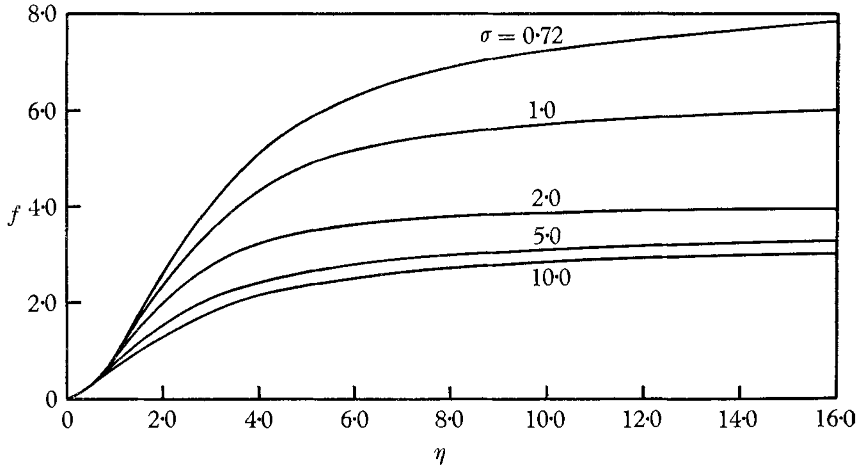

FIGURE 5. Axisymmetric jet. f *vs.* η for several values of σ.

Exact solutions for the system (5.15), (5.16), are possible for certain Prandtl numbers. If one assumes

$$f = b\eta^2/(b+\eta^2), \tag{5.19}$$

then integration of (5.16) provides p:

$$p = c(b+\eta^2)^{-\frac{1}{2}\sigma b}. \tag{5.20}$$

Formulas (5.19) and (5.20) satisfy equation (5.15) only for the following choices of σ, b and c:

case (i) $\sigma = 1, \quad b = 6, \quad c = 576;$

case (ii) $\sigma = 2, \quad b = 4, \quad c = 1{,}024.$

Numerical integration of (5.15) and (5.16) for several other values of the Prandtl number provides the family of solutions shown in figures 5–7.

From the solutions for f and p, one may compute the velocity components and the temperature from equations (5.17), (5.18) and (5.11). The constant a, and the location of the origin of x can be chosen in an identical way with that of the two-dimensional jet. Suppose that at some x, say $x = x_0$, the velocity and temperature profiles are measured. Then if W is the measured volume flow rate and E is the integral of the measured temperature distribution, one can select a and x_0 so as to make the corresponding integrals of the mathematical solution equal to the measured quantities:

$$2\pi \int_0^\infty u(x_0, y) y\, dy = W, \tag{5.21}$$

$$2\pi \int_0^\infty [T(x_0, y) - T_\infty] y\, dy = E. \tag{5.22}$$

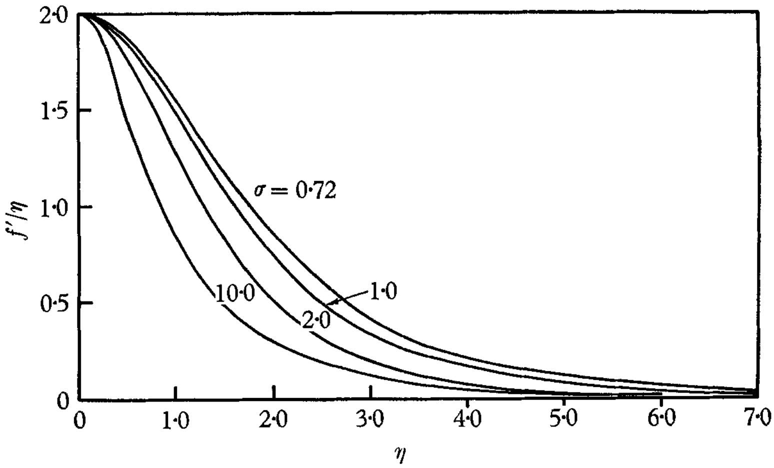

FIGURE 6. Axisymmetric jet. f'/η *vs.* η for several values of σ.

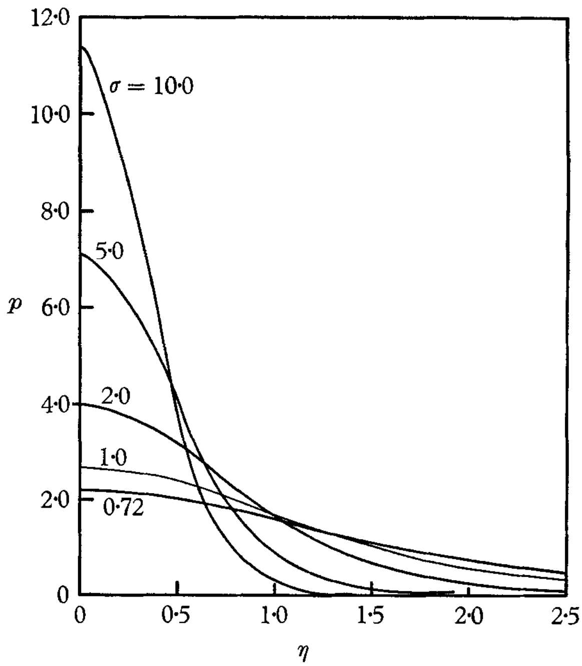

FIGURE 7. Axisymmetric jet. p *vs.* η for several values of σ.

When equations (5.7) and (5.17) are substituted into (5.21) and (5.22), the resulting equations may be solved for a and x_0,

$$a^2 = \frac{g\beta E f(\infty)}{2\pi\nu^3 W \int_0^\infty \eta p \, d\eta}, \tag{5.23}$$

$$x_0^2 = \frac{W}{2\pi\nu f(\infty)}. \tag{5.24}$$

The value of x_0 now locates the origin of the co-ordinate system, and the mathematical solutions may be assumed to represent the actual jet flow for $x \geqslant x_0$.

Values of $f(\infty)$ and the integral in (5.23) are presented in table 2 for several Prandtl numbers.

σ	$f(\infty)$	$\int_0^\infty p\eta \, d\eta$
0·72	8·062	4·937
1·0	6·000	3·852
2·0	4·000	2·667
5·0	3·375	1·738
10·0	3·143	1·125

TABLE 2

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